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Three general multivariate semi-Pareto distributions and their characterizations

Hsiaw-Chan Yeh*

Department of Finance, College of Management, National Taiwan University, Rm. 1002, No.85, Roosevelt Road, Section 4, Taipei, Taiwan, 106 ROC

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Abstract

Three general multivariate semi-Pareto distributions are developed in this paper. First one—GMP$^{(k)}$ (III) has univariate Pareto (III) marginals, it is characterized by the minimum of two independent and identically distributed random vectors. Second one—GMSP has univariate semi-Pareto marginals and it is characterized by finite sample minima. Third one—MSP is characterized through a geometric minimization procedure. All these three characterizations are based on the general and the particular solutions of the Euler's functional equations of k-variates.

AMS 1991 subject classification: 62E10

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1. Introduction and motivation

It is well known that the multivariate Pareto distributions fit good to the upper tails of some multivariate continuous income data and some other socio-economic multivariate variables.

Numerous papers dealing with various bivariate and multivariate Pareto distributions have subsequently appeared in the literature after Arnold [2] (see [7, Chapter 52] and the references therein). Three more general multivariate Pareto (III) distributions than Arnold [2] MP$^{(k)}$ (III) are developed in this paper. The first one denoted by GMP$^{(k)}$ (III) and is in Section 2. A characterization

* Fax: 886 2 2366 0764.
E-mail address: Yeh12345@management.ntu.edu.tw.

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of the $\mathrm{GMP}^{(k)}$(III) obtained through the distribution of the minimum of two independent and identically distributed (i.i.d.) random vectors is proved in Section 2.

The second general multivariate Pareto distribution denoted by $\mathrm{GMSP}^{(k)}$ is a new and more general multivariate semi-Pareto distribution with univariate semi-Pareto marginals. The univariate semi-Pareto distribution was introduced by Pillai [8]. Some special bivariate semi-Pareto distributions were researched by Balakrishna and Jayakumar [3]. The more general multivariate semi-Pareto distribution is developed and its characterization is studied in Section 3.

The third general multivariate Pareto denoted by $\mathrm{MSP}^{(k)}$ is introduced in Section 4. Balakrishna and Jayakumar [3] studied some specific bivariate semi-Pareto distribution with homogeneous scale parameters, but they did not mention too much about the multivariate cases. Yeh [11] had studied the characterization of multivariate $\mathrm{MP}^{(k)}$(III) discussed by Arnold [2], through the geometric minimization procedures. All of these results can be extended to the MSP distribution and are studied in Section 4.

The relationships among the three general multivariate Pareto (III) distributions developed in this paper as well as Arnold [2] $\mathrm{MP}^{(k)}$(III) are depicted as the following diagram:

$$\mathrm{MP}^{(k)}(\mathrm{III}) \subset \mathrm{GMP}^{(k)}(\mathrm{III}) \subset \mathrm{GMSP}^{(k)}, \text{ or equivalently, Eqs. (2.1)} \subset (2.2) \subset (3.1),$$

and

$$\mathrm{MP}^{(k)}(\mathrm{III}) \subset \mathrm{MSP}^{(k)} \subset \mathrm{GMSP}^{(k)}, \text{ or equivalently, Eqs. (2.1)} \subset (4.1) \subset (3.1).$$

The technical proofs of all the characterization theorems in this paper are based on the general and the particular solutions of the Euler's functional equations of $k(\geqslant 1)$ variables, they are new to the existent references.

2. The generalized multivariate Pareto (III): $\mathrm{GMP}^{(k)}$ distribution

According to Arnold [2]'s definition, the third type of the multivariate Pareto, $\mathrm{MP}^{(k)}$(III) distribution is defined as follows:

Definition 2.1. A k-variate random vector $\underline{X} = (X_1, \ldots, X_k)$ is said to follow a $\mathrm{MP}^{(k)}$(III) $(\underline{\sigma}, \underline{\gamma})$ distribution, if its joint survival function is of the form

$$\overline{F}_{\underline{X}}(\underline{x}) = \left\{ 1 + \sum_{i=1}^{k} \left(\frac{x_i}{\sigma_i} \right)^{1/\gamma_i} \right\}^{-1} \tag{2.1}$$

for any $\underline{x} = (x_1, \ldots, x_k) \geqslant \underline{0}$, where $\underline{\sigma} = (\sigma_1, \cdots, \sigma_k) > \underline{0}$ and $\underline{\gamma} = (\gamma_1, \cdots, \gamma_k) > \underline{0}$ are scale and shape parameters, respectively, and it is denoted by $\underline{X} \sim \mathrm{MP}^{(k)}(\mathrm{III})\ (\underline{\sigma}, \underline{\gamma})$.

A straightforward extension of the $\mathrm{MP}^{(k)}$(III) is the following:

Definition 2.2. A random vector $\underline{X} = (X_1, \cdots, X_k)$ is said to follow a generalized $\mathrm{MP}^{(k)}$(III) distribution and is denoted by $\mathrm{GMP}^{(k)}$(III) $(\underline{\sigma}, \underline{\gamma})$, if its k marginals of each X_i in $\underline{X}$ are univariate Pareto(III) (σ_i, γ_i) distributed, i.e., the survival function of each X_i is of the form

$$\overline{F}_i(x_i) = \left\{ 1 + \left(\frac{x_i}{\sigma_i} \right)^{1/\gamma_i} \right\}^{-1} \tag{2.2}$$

for any $x_i > 0$ and $\sigma_i > 0$, $\gamma_i > 0$.

Clearly, $\mathrm{MP}^{(k)}(\mathrm{III})\ (\underline{\sigma}, \underline{\gamma}) \subset \mathrm{GMP}^{(k)}(\mathrm{III})\ (\underline{\sigma}, \underline{\gamma})$. The following theorem characterizes the $\mathrm{GMP}^{(k)}(\mathrm{III})$ distribution.

Theorem 2.1. *Let* $\underline{X}^1 = (X_1^1, X_2^1, \cdots, X_k^1)$ *and* $\underline{X}^2 = (X_1^2, X_2^2, \cdots, X_k^2)$ *be two k-variate independent random vectors with support on* $(0, \infty)^k$ *with common joint survival function* $\overline{F}_{\underline{X}}(\cdot)$, *and let* p_1, p_2 *be two constants such that* $p_1^\alpha + p_2^\alpha = 1$, $0 < p_1, p_2 < 1$, $\alpha > 0$. *Then the following two statements are equivalent*:

(1) $$\min\left(\frac{1}{p_1}\underline{X}^1, \frac{1}{p_2}\underline{X}^2\right) \overset{d}{=} \min\left(\underline{X}^1, \max\left(\frac{1}{p_1}\underline{Y}^1, \quad \frac{1}{p_2}\underline{Y}^2\right)\right), \tag{2.3}$$

where $(\underline{Y}^1, \underline{Y}^2)$ *is an independent copy of* $(\underline{X}^1, \underline{X}^2)$, *and the minimum and maximum are defined componentwisely as*

$$\min\left(\frac{1}{p_1}\underline{X}^1, \frac{1}{p_2}\underline{X}^2\right) \overset{\Delta}{=} \left(\min\left\{\frac{1}{p_1}X_1^1, \frac{1}{p_2}X_1^2\right\}, \min\left\{\frac{1}{p_1}X_2^1, \frac{1}{p_2}X_2^2\right\}, \cdots, \right.$$
$$\left. \min\left\{\frac{1}{p_1}X_k^1, \frac{1}{p_2}X_k^2\right\}\right)$$

and it is a k-variate random vector. On the other hand, the RHS of Eq. (2.3) *is defined componentwisely as*

$$\min\left(\underline{X}^1, \max\left(\frac{1}{p_1}\underline{Y}^1, \frac{1}{p_2}\underline{Y}^2\right)\right)$$
$$\overset{\Delta}{=} \left(\min\left\{X_1^1, \max\left\{\frac{1}{p_1}Y_1^1, \frac{1}{p_2}Y_1^2\right\}\right\}, \min\left\{X_2^1, \max\left\{\frac{1}{p_1}Y_2^1, \frac{1}{p_2}Y_2^2\right\}\right\}, \cdots, \right.$$
$$\left. \min\left\{X_k^1, \max\left\{\frac{1}{p_1}Y_k^1, \frac{1}{p_2}Y_k^2\right\}\right\}\right)$$

is also a k-variate random vector.

(2) *The common joint survival function* $\overline{F}_{\underline{X}}(\cdot)$ *is a general multivariate Pareto* (III) *with univariate Pareto* (III) (α, σ_i) *as marginals for some* $\sigma_i > 0$, $i = 1, 2, \cdots, k$, *i.e., the survival function for each* X_i in $\underline{X}$ is $\overline{F}_i(x_i) = \left\{1 + \left(\frac{x_i}{\sigma_i}\right)^\alpha\right\}^{-1}, \quad \forall x_i > 0.$

Proof. (1) $\Rightarrow$ (2). Suppose Eq. (2.3) holds, then for each component i, $i = 1, 2, \cdots, k$,

$$\min\left(\frac{1}{p_1}X_i^1, \frac{1}{p_2}X_i^2\right) \overset{d}{=} \min\left(X_i^1, \max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\}\right).$$

Then, let $\overline{F}_i(\cdot)$ be the ith marginal survival function of the common survival function of $\underline{X}^1, \underline{X}^2$, then for any x_i in $\underline{x} = (x_1, \cdots, x_k) > \underline{0}$,

$$P\left(\min\left(\frac{1}{p_1}X_i^1, \frac{1}{p_2}X_i^2\right) > x_i\right) = P\left(\min\left(X_i^1, \max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\} > x_i\right)\right),$$

$$\Rightarrow \quad P(X_i^1 > p_1 x_i)P(X_i^2 > p_2 x_i) = P(X_i^1 > x_i)P\left(\max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\} > x_i\right),$$

i.e.,

$$\overline{F}_i(p_1 x_i)\overline{F}_i(p_2 x_i) = \overline{F}_i(x_i)\{1 - F_i(p_1 x_i)F_i(p_2 x_i)\} \tag{2.4}$$

for each marginal, let $\psi_i(x_i) = \frac{1-\overline{F}_i(x_i)}{\overline{F}_i(x_i)}$, so $\overline{F}_i(x_i) = \frac{1}{1+\psi_i(x_i)}$, then Eq. (2.4) becomes

$$\left(\frac{1}{1+\psi_i(p_1x_i)}\right)\cdot\left(\frac{1}{1+\psi_i(p_2x_i)}\right) = \left(\frac{1}{1+\psi_i(x_i)}\right) \times\left\{1-\frac{\psi_i(p_1x_i)}{1+\psi_i(p_1x_i)}\cdot\frac{\psi_i(p_2x_i)}{1+\psi_i(p_2x_i)}\right\},$$

thus, $\psi_i(x_i) = \psi_i(p_1x_i) + \psi_i(p_2x_i)$, for any $p_1^\alpha + p_2^\alpha=1$, $0 < p_1, p_2 < 1$, $\alpha > 0$.

Therefore, by Castillo and Ruiz–Cobo [4], the particular solution for the generalized Cauchy functional equation is $\psi_i(x_i) = \left(\frac{x_i}{\sigma_i}\right)^\alpha$ for some $\sigma_i > 0$, so the marginal survival function of $\underline{X}$ is $\overline{F}_i(x_i) = \left\{1+\left(\frac{x_i}{\sigma_i}\right)^\alpha\right\}^{-1}$, i.e., $X_i \sim$ univariate Pareto $P(\mathrm{III})$ (α, σ_i), $i = 1, \cdots, k$. Then (1) implies (2).

Conversely, (2) $\Rightarrow$ (1). Suppose the k marginals of the common joint survival function $\overline{F}_{\underline{X}}(\cdot)$ are univariate $P(\mathrm{III})(\alpha, \sigma_i)$, for $i = 1, 2, \cdots, k$, i.e., $\overline{F}_i(x_i) = \left\{1+\left(\frac{x_i}{\sigma_i}\right)^\alpha\right\}^{-1}$, $x_i > 0$, the survival function of $\min\left(\frac{1}{p_1}\underline{X}^1, \frac{1}{p_2}\underline{X}^2\right)$ for any $\underline{x} > \underline{0}$ is

$$\begin{aligned} P\left(\min\left(\frac{1}{p_1}\underline{X}^1, \frac{1}{p_2}\underline{X}^2\right) > \underline{x}\right) &= P(\underline{X}^1 > p_1\underline{x})P(\underline{X}^2 > p_2\underline{x}) \\ &= \overline{F}_{\underline{X}}(p_1\underline{x})\cdot\overline{F}_{\underline{X}}(p_2\underline{x}). \end{aligned} \tag{2.5}$$

On the other hand, the survival function of $\min\left(\underline{X}^1, \max\left(\frac{1}{p_1}\underline{Y}^1, \frac{1}{p_2}\underline{Y}^2\right)\right)$ is

$$\begin{aligned} &P\left(\min\left\{\underline{X}^1, \max\left\{\frac{1}{p_1}\underline{Y}^1, \frac{1}{p_2}\underline{Y}^2\right\}\right\} > \underline{x}\right) \\ &= P(\underline{X}^1 > \underline{x})P\left(\max\left\{\frac{1}{p_1}\underline{Y}^1, \frac{1}{p_2}\underline{Y}^2\right\} > \underline{x}\right) \\ &= \overline{F}_{\underline{X}}(x)\cdot P\left(\max\left\{\frac{1}{p_1}\underline{Y}^1, \frac{1}{p_2}\underline{Y}^2\right\} > \underline{x}\right). \end{aligned} \tag{2.6}$$

Consider the ith marginal of Eq. (2.5) which is

$$\begin{aligned} P\left(\min\left\{\frac{1}{p_1}X_i^1, \frac{1}{p_2}X_i^2\right\} > x_i\right) &= \overline{F}_i(p_1x_i)\cdot\overline{F}_i(p_2x_i) \\ &= \left(\frac{1}{1+p_1^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right)\cdot\left(\frac{1}{1+p_2^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right), \end{aligned} \tag{2.7}$$

and the ith marginal of Eq. (2.6) is

$$\begin{aligned} &P\left(\min\left(X_i^1, \max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\}\right) > x_i\right) \\ &= \overline{F}_i(x_i)\cdot P\left(\max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\} > x_i\right). \end{aligned} \tag{2.8}$$

Note that

$$P\left(\max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\} > x_i\right) = 1 - P\left(\max\left\{\frac{1}{p_1}Y_i^1, \frac{1}{p_2}Y_i^2\right\} \leqslant x_i\right)$$
$$= 1 - F_i(p_1x_i)F_i(p_2x_i)$$
$$= 1 - \left(\frac{p_1^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}{1+p_1^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right)\cdot\left(\frac{p_2^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}{1+p_2^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right).$$

Comparing two equations (2.7) and (2.8),

$$\left(\frac{1}{1+p_1^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right)\left(\frac{1}{1+p_2^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right)$$
$$= \frac{1}{1+\left(\frac{x_i}{\sigma_i}\right)^\alpha}\cdot\left\{1-\left(\frac{p_1^\alpha(\frac{x_i}{\sigma_i})^\alpha}{1+p_1^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right)\left(\frac{p_2^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}{1+p_2^\alpha\left(\frac{x_i}{\sigma_i}\right)^\alpha}\right)\right\}.$$

They are indeed the same, hence componentwisely,

$$\min\left(\frac{1}{p_1}X_i^1, \frac{1}{p_2}X_i^2\right) \stackrel{\mathrm{d}}{=} \min\left(X_i^1, \max\left(\frac{1}{p_1}Y_i^1, \quad \frac{1}{p_2}Y_i^2\right)\right)$$

for each $i = 1, 2, \cdots, k$, thus the joint distribution of this two k-variate random vectors min $\left(\frac{1}{p_1}\underline{X}^1, \frac{1}{p_2}\underline{X}^2\right)$ and min $\left(\underline{X}^1, \max\left(\frac{1}{p_1}\underline{Y}^1, \frac{1}{p_2}\underline{Y}^2\right)\right)$ are marginally equivalent.

This completes the proof. $\square$

3. The generalized multivariate semi-Pareto distribution

A broad class of multivariate semi-Pareto distribution is introduced and characterized in this section.

Definition 3.1. A random vector $\underline{X} = (X_1, X_2, \cdots, X_k)$ is said to follow a generalized multivariate semi-Pareto distribution if its k marginals of each X_i in $\underline{X}$ are univariate semi-Pareto $SP(\alpha_i, \underline{p} = (p_1, p_2, \cdots, p_\ell))$, i.e., the survival function of each X_i is of the form

$$\overline{F}_i(x_i) = \frac{1}{1+\psi_i(x_i)}, \tag{3.1}$$

where $\psi_i(x_i)$ satisfies the functional equation

$$\psi_i(x_i) = \sum_{j=1}^{\ell} p_j\psi_i\,(p_j^{-1/\alpha_i}x_i) \tag{3.2}$$

for any $x_i > 0$, and some $\alpha_i > 0$, $i = 1, 2, \cdots, k$, and $0 < p_j < 1$, $j = 1, 2, \cdots, \ell$, and $\ell \in \mathbf{N}$ is a positive integer, $\ell \geqslant 2$, and such a random vector $\underline{X}$ is denoted by $\underline{X} \sim \mathrm{GMSP}^{(k)}(\underline{\alpha}, \underline{p})$, where $\underline{\alpha} = (\alpha_1, \cdots, \alpha_k)$, and $\underline{p} = (p_1, \cdots, p_\ell)$.

Definition 3.1 is also available in Jaykumar [6].

The following theorem characterizes this generalized multivariate semi-Pareto distribution.

Theorem 3.1. *Let* $\{\underline{X}^j = (X_1^j, X_2^j, \cdots, X_k^j)\}_{j=1}^{\ell}$ be $\ell(\geqslant 2)$'*s k-variate independent random vectors with support on* $(0,\infty)^k$ *and common joint survival function,* $\overline{F}_{\underline{X}}(\cdot)$, *and let* b_j, $j = 1, 2, \cdots, \ell$, *be* ℓ *constants such that* $b_j > 0$, *and for each* j, *let the k-variate random vector be defined as*

$$\underline{Y}^j = \min_{\substack{1\leqslant h\leqslant \ell\\ h\neq j}}\{b_h^{-1}\cdot \underline{X}^h\} \stackrel{\Delta}{=} (Y_1^j, Y_2^j, \cdots, Y_k^j),$$

the ith coordinate of $\underline{Y}^j$ is Y_i^j, $i = 1, 2, \cdots, k$, *is defined as*

$$Y_i^j = \min\{b_1^{-1}X_i^1, b_2^{-1}X_i^2, \cdots, b_{j-1}^{-1}X_i^{j-1}, b_{j+1}^{-1}X_i^{j+1}, \cdots, b_\ell^{-1}X_i^\ell\}, \tag{3.3}$$

and let a_j, $j = 1, 2, \cdots, \ell$, *satisfy* $\sum_{j=1}^{\ell} a_j = 1$. *Then the following two statements are equivalent*:

(1) *The ith marginal survival function of* $\overline{F}_{\underline{X}}(\cdot)$ *satisfies the functional equation*

$$\overline{F}_i(x_i) = \frac{\prod_{j=1}^{\ell}\overline{F}_i(b_jx_i)}{\sum_{j=1}^{\ell} a_j \overline{F}_{Y_i^j}(x_i)}. \tag{3.4}$$

(2) *The common joint survival function* $\overline{F}_{\underline{X}}(\cdot)$ *is a GMSP*$^{(k)}(\underline{\alpha}, \underline{p})$ *distribution defined as in Definition* 3.1.

Proof. (1) $\Rightarrow$ (2). Suppose Eq. (3.4) holds, i.e.,

$$\overline{F}_i(x_i) = \frac{\prod_{j=1}^{\ell}\overline{F}_i(b_jx_i)}{\sum_{j=1}^{\ell} a_j \overline{F}_{Y_i^j}(x_i)},$$

where

$$\overline{F}_{Y_i^j}(x_i) = P\left(\min_{\substack{1\leqslant h\leqslant \ell\\ h\neq j}} b_h^{-1}X_i^h > x_i\right) = \prod_{\substack{1\leqslant h\leqslant \ell\\ h\neq j}} \overline{F}_i(b_hx_i), \tag{3.5}$$

then $\overline{F}_i(x_i)$ becomes

$$\overline{F}_i(x_i) = \frac{\prod_{j=1}^{\ell}\overline{F}_i(b_jx_i)}{\sum_{j=1}^{\ell} a_j\left(\prod_{\substack{1\leqslant h\leqslant \ell\\ h\neq j}} \overline{F}_i(b_hx_i)\right)} = \frac{1}{\sum_{j=1}^{\ell} a_j\left(\frac{1}{\overline{F}_i(b_jx_i)}\right)}, \tag{3.6}$$

express each marginal by writing $\psi_i(x_i) = \frac{1-\overline{F}_i(x_i)}{\overline{F}_i(x_i)}$, so $\overline{F}_i(x_i) = \frac{1}{1+\psi_i(x_i)}$, then Eq. (3.6) becomes

$$\begin{aligned}\frac{1}{1+\psi_i(x_i)} &= \frac{1}{\sum_{j=1}^{\ell} a_j(1+\psi_i(b_jx_i))} = \frac{1}{\sum_{j=1}^{\ell} a_j + \sum_{j=1}^{\ell} a_j\psi_i(b_jx_i)}\\ &= \frac{1}{1+\sum_{j=1}^{\ell} a_j\psi_i(b_jx_i)},\end{aligned} \tag{3.7}$$

thus, $\psi_i(x_i) = \sum_{j=1}^{\ell} a_j \psi_i(b_j x_i)$, if b_j is chosen to be $b_j = a_j^{-1/\alpha_i}$, for some $\alpha_i > 0$, then the common joint survival function $\overline{F}_{\underline{X}}(\cdot)$ is a generalized multivariate semi-Pareto distribution given in Definition 3.1, and therefore (2) follows.

(2)⇒(1). Conversely, suppose the k marginals of the common joint survival function $\overline{F}_{\underline{X}}(\cdot)$ are univariate semi-Pareto SP(III) $(\alpha_i, \underline{p} = (p_1, \cdots, p_\ell))$, i.e., the function $\overline{F}_i(x_i) = \frac{1}{1+\psi_i(x_i)}$, satisfying the functional equation $\psi_i(x_i) = \sum_{j=1}^{\ell} p_j \psi_i(p_j^{-1/\alpha_i} x_i)$, if choose $a_j \equiv p_j$ with $\sum_{j=1}^{\ell} a_j = 1$, and $b_j \equiv p_j^{-1/\alpha_i}$, then Eq. (3.7) holds, and thus from Eq. (3.6), we get

$$\prod_{j=1}^{\ell} \overline{F}_i(p_j^{-1/\alpha_i} x_i) = \overline{F}_i(x_i) \cdot \left\{ \sum_{j=1}^{\ell} p_j \left(\prod_{\substack{1 \leqslant h \leqslant \ell \\ h \neq j}} (\overline{F}_i(p_h^{-1/\alpha_i} x_i)) \right) \right\}. \tag{3.8}$$

From Eqs. (3.8) and (3.5) it is clear that Eq. (3.4) is true, hence (1) is followed. □

4. Multivariate semi-Pareto distributions

A more general k-variate semi-Pareto distribution than those proposed by Balakrishna and Jayakumar [3] as well as Thomas and Jose [9,10] is given below:

Definition 4.1. The random vector $\underline{X} = (X_1, X_2, \cdots, X_k)$ is said to have a k-variate semi-Pareto distribution with parameters, $p \in (0, 1)$, $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k) > \underline{0}$ and the scale parameter, $\underline{\sigma} = (\sigma_1, \cdots, \sigma_k) > \underline{0}$, and $\underline{X}$ is denoted by $\underline{X} \sim \mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$, if its survival function is of the form

$$\overline{F}_{\underline{X}}(\underline{x}) = \frac{1}{1 + \Psi(x_1, \cdots, x_k)}$$

such that

$$\Psi(x_1, x_2, \cdots, x_k) = \frac{1}{p} \cdot \Psi(p^{1/\alpha_1} x_1, p^{1/\alpha_2} x_2, \cdots, p^{1/\alpha_k} x_k) \tag{4.1}$$

The solution of Eq. (4.1) is mentioned in Aczél [1, p. 231].

Before we come to the characterization of the k-variate semi-Pareto distribution, there is one lemma concerning the general solution of Euler's (1755, 1768, 1770) [5] (cited in Aczél [1]) functional equation for homogeneous functions will be introduced as below:

Lemma 4.1. *The general solution of the k ($\geqslant 2$) variables in Euler's functional equation*

$$F(px_1, px_2, \cdots, px_k) = p^{\alpha} F(x_1, x_2, \cdots, x_k) \tag{4.2}$$

for $\alpha \neq 0$, $p \neq 0$, and $x_1 \neq 0$, is

$$F(x_1, x_2, \cdots, x_k) = x_1^{\alpha} \cdot f\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \cdots, \frac{x_k}{x_1}\right), \tag{4.3}$$

where $f(\cdot)$ is an arbitrary function of $(k-1)$ variables in terms of $\{x_2/x_1, \cdots, x_k/x_1\}$.

Proof. The k-variate ($k \geqslant 2$) Euler's functional equation is solved as

$$\begin{aligned} F(x_1, x_2, \cdots, x_k) &= F\left(x_1 \cdot 1, x_1 \cdot \frac{x_2}{x_1}, \cdots, x_1 \cdot \frac{x_k}{x_1}\right) \\ &= x_1^{\alpha} \cdot F\left(1, \frac{x_2}{x_1}, \cdots, \frac{x_k}{x_1}\right) \underline{\underline{\Delta}} x_1^{\alpha} \cdot f\left(\frac{x_2}{x_1}, \cdots, \frac{x_k}{x_1}\right), \end{aligned} \tag{4.4}$$

where $f(\cdot)$ is an arbitrary function of $(k-1)$ variables in terms of $\{x_2/x_1, \cdots, x_k/x_1\}$. and does satisfy Eq. (4.2) for $x_1 \neq 0$, $\alpha \neq 0$. Hence, Eq. (4.3) is the general solution of (4.2), and thus this lemma follows. □

Lemma 4.1 can be extended to the so-called almost homogeneous functions

$$F(p^{\ell_1} x_1, p^{\ell_2} x_2, \cdots, p^{\ell_k} x_k) = p^{\ell} F(x_1, x_2, \cdots, x_k) \tag{4.5}$$

for $\ell \neq 0$, $\ell_i \neq 0$, $i = 1, 2, \cdots, k$.

For the sake of simplicity, we consider only the positive real values of $p > 0$ and $x_i > 0$, $i = 1, 2, \cdots, k$.

Lemma 4.2. *The general solution of the k* ($\geqslant 2$) *variables in the functional equations of the form* (4.5) *is*

$$F(x_1, x_2, \cdots, x_k) = x_1^{\ell/\ell_1} \cdot f\left(\frac{x_2}{x_1^{\ell_2/\ell_1}}, \frac{x_3}{x_1^{\ell_3/\ell_1}}, \cdots, \frac{x_k}{x_1^{\ell_k/\ell_1}}\right), \tag{4.6}$$

where $f(\cdot)$ *is an arbitrary function of* $(k-1)$ *positive variables in terms of* $\{x_2/x_1, x_3/x_1, \cdots, x_k/x_1\}$, *and* $\ell \neq 0$, $\ell_i \neq 0$, *and* $x_i > 0$, $i = 1, 2, \cdots, k$.

Proof. Choose $p = x_1^{-1/\ell_1}$ and plug in Eq. (4.5), then Eq. (4.5) becomes

$$F((x_1^{-1/\ell_1})^{\ell_1} x_1, \quad (x_1^{-1/\ell_1})^{\ell_2} x_2, \cdots, (x_1^{-1/\ell_1})^{\ell_k} x_k) = (x_1^{-1/\ell_1})^{\ell} F(x_1, \cdots, x_k).$$

This implies $F\left(1, \frac{x_2}{x_1^{\ell_2/\ell_1}}, \cdots, \frac{x_k}{x_1^{\ell_k/\ell_1}}\right) = x_1^{-\ell/\ell_1} \cdot F(x_1, \cdots, x_k)$.

Thus, a direct and general solution of Eq. (4.5) is

$$\begin{aligned} F(x_1, \cdots, x_k) &= x_1^{\ell/\ell_1} \cdot F\left(1, \frac{x_2}{x_1^{\ell_2/\ell_1}}, \cdots, \frac{x_k}{x_1^{\ell_k/\ell_1}}\right) \\ &= x_1^{\ell/\ell_1} \cdot f\left(\frac{x_2}{x_1^{\ell_2/\ell_1}}, \cdots, \frac{x_k}{x_1^{\ell_k/\ell_1}}\right), \end{aligned}$$

where $f(\cdot)$ is an arbitrary function of $(k-1)$ positive variables in terms of $\{x_2/x_1, x_3/x_1, \cdots, x_k/x_1\}$. A special case of $\ell = 1$ in Eq. (4.5) is considered in the following corollary.

Corollary 4.2.1. *The particular solution of the functional equation*

$$F(p^{\ell_1}x_1, p^{\ell_2}x_2, \cdots, p^{\ell_k}x_k) = pF(x_1, \cdots, x_k) \tag{4.7}$$

is either

$$\text{(i)} \quad F(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k} x_i^{1/\ell_i}, \tag{4.8}$$

or

$$\text{(ii)} \quad F(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k} \left(\frac{x_i}{\sigma_i}\right)^{1/\ell_i} \tag{4.9}$$

for some $\underline{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_k) > \underline{0}$.

Proof. Choose the arbitrary function $f(\cdot)$ in Eq. (4.6) as

$$\text{(i)} \quad f\left(\frac{x_2}{x_1^{\ell_2/\ell_1}}, \cdots, \frac{x_k}{x_1^{\ell_k/\ell_1}}\right) = 1 + \sum_{i=2}^{k}\left(\frac{x_i}{x_1^{\ell_i/\ell_1}}\right)^{1/\ell_i} = 1 + \frac{\sum_{i=2}^{k} x_i^{1/\ell_i}}{x_1^{1/\ell_1}}.$$

Hence, the particular solution of Eq. (4.6) is

$$F(x_1, x_2, \cdots, x_k) = x_1^{1/\ell_1} \cdot \left\{1 + \frac{\sum_{i=2}^{k} x_i^{1/\ell_i}}{x_1^{1/\ell_1}}\right\} = \sum_{i=1}^{k} x_i^{1/\ell_i}.$$

(ii) Consider the scale transformation on each variable as x_i/σ_i, for $\sigma_i > 0$, $i = 1, 2, \cdots, k$. It is straightforward to check that Eq. (4.9) does satisfy Eq. (4.7) as

$$\begin{aligned} F(p^{\ell_1}x_1, p^{\ell_2}x_2, \cdots, p^{\ell_k}x_k) &= \sum_{i=1}^{k}\left(\frac{p^{\ell_i}x_i}{\sigma_i}\right)^{1/\ell_i} = p\sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{1/\ell_i} \\ &= p \cdot F(x_1, \cdots, x_k). \end{aligned}$$

Thus, this corollary is followed. □

Referring back to Definition 4.1, Eq. (4.1), it is clear that $\Psi(x_1, x_2, \cdots, x_k) = \sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{\alpha_i}$ is the particular solution of Eqs. (4.1) and (4.2), and the k-variate semi-Pareto $\text{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$ reduces to the k-variate Arnold [2] $\text{MP}^{(k)}(\text{III})$ $(\underline{0}, \underline{\sigma}, \underline{\alpha})$ distribution with joint survival function

$$\overline{F}_{\underline{X}}(x_1, x_2, \cdots, x_k) = \frac{1}{1 + \sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{\alpha_i}} \tag{4.10}$$

for $\underline{x} = (x_1, \cdots, x_k) > \underline{0}$.

Balakrishna and Jayakumar [3] had studied the characterization of the bivariate semi- Pareto distribution for the homogeneous scales $\sigma_i \equiv 1$ case, and Yeh [11] also studied the characterization of the Arnold [2] $\text{MP}^{(k)}(\text{III})$ $(\underline{0}, \underline{\sigma}, \underline{\alpha})$ distribution through a geometric minimization procedure. All these results can be extended to the following k ($\geqslant 2$) variate semi-Pareto distribution $\mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$ with joint survival function of the form in Eq. (4.1) and the parameter p is restricted in $0 < p < 1$, and $\underline{\alpha} > \underline{0}$, $\underline{\sigma} > \underline{0}$.

Theorem 4.1. *Let* $\{\underline{X}^i = (X_1^i, X_2^i, \cdots, X_k^i)\}_1^n$ *be a sequence of i.i.d. nonnegative random vectors with common joint survival function* $\overline{F}_{\underline{X}}(\cdot)$. *For a fixed* $p \in (0, 1)$, *let* $N_p \sim geometric(p)$, N_p *is independent of all* $\underline{X}^i$, *and let* $\underline{m}_p = (X_{(1)}, \cdots, X_{(k)})$ *be the k-dimensional geometric minima with* $X_{(j)} = \min\{X_j^1, X_j^2, \cdots, X_j^N\}$ *for each* $j = 1, 2, \cdots, k$, *then*

$$p^{-\underline{\alpha}^{-1}} \underline{m}_p \underline{\underline{\Delta}} (p_1^{-1/\alpha_1} X_{(1)}, p^{-1/\alpha_2} X_{(2)}, \cdots, p^{-1/\alpha_k} X_{(k)}) \stackrel{d}{=} \underline{X}^1$$

if and only if $\underline{X}^1 \sim \mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$, *where the parameter vector* $-\underline{\alpha}^{-1} = (-\alpha_1^{-1}, -\alpha_2^{-1}, \cdots, -\alpha_k^{-1})$.

Proof. Let $\overline{H}_{\underline{m}_p}(\cdot)$ be the joint survival function of $p^{-\underline{\alpha}^{-1}} \cdot \underline{m}_p$, i.e.,

$$\begin{aligned}
\overline{H}_{\underline{m}_p}(\underline{x}) &= P(p^{-\underline{\alpha}^{-1}} \cdot \underline{m}_p \geqslant \underline{x}) = P(X_{(1)} > p^{1/\alpha_1} x_1, X_{(2)} > p^{1/\alpha_2} x_2, \cdots, X_{(k)} > p^{1/\alpha_k} x_k) \\
&= \sum_{n=1}^{\infty} P(\underline{X}^1 \geqslant p^{\underline{\alpha}^{-1}} \underline{x})^n \cdot p(1-p)^{n-1} = \frac{p\overline{F}_{\underline{X}}(p^{\underline{\alpha}^{-1}} \underline{x})}{1 - (1-p)\overline{F}_{\underline{X}}(p^{\underline{\alpha}^{-1}} \underline{x})} \\
&= \overline{F}_{\underline{X}}(\underline{x}),
\end{aligned} \tag{4.11}$$

where $\underline{x} = (x_1, \cdots, x_k) > \underline{0}$ and $p^{\underline{\alpha}^{-1}} \cdot \underline{x} = (p^{1/\alpha_1} x_1, \cdots, p^{1/\alpha} x_k)$. Hence, the common joint survival function of each $\underline{X}^i$, $\overline{F}_{\underline{X}}(\cdot)$ satisfies the functional equation.

Let $\varphi(\underline{x}) = \frac{1-\overline{F}_{\underline{X}}(\underline{x})}{\overline{F}_{\underline{X}}(\underline{x})}$, so $\overline{F}_{\underline{X}}(\underline{x}) = \frac{1}{1+\varphi(\underline{x})}$, substitute $\varphi(\cdot)$ in (4.11), we conclude that for all $\underline{x} > \underline{0}$, we have $\frac{1}{1+\varphi(\underline{x})} = \frac{p}{p+\varphi(p^{\underline{\alpha}^{-1}} \underline{x})}$, then the functional equation $\varphi(\cdot)$ satisfies $\varphi(\underline{x}) = \frac{1}{p}\varphi(p^{\underline{\alpha}^{-1}} \underline{x})$. This is the functional Eq. (4.1) satisfied by $\mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$ with $\varphi(\cdot)$ in place of $\psi(\cdot)$, hence $\underline{X}^1 \sim \mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$ is followed.

On the other hand, if $\underline{X}^i \sim \mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$, then its survival function is of the form (4.1) and let $\overline{H}_{\underline{m}_p}(\cdot)$ be the joint survival function of $p^{-\underline{\alpha}^{-1}} \cdot \underline{m}_p$. It is derived as in Eq. (4.11) by conditioning on N_p,

$$\overline{H}_{\underline{m}_p}(\underline{x}) = \sum_{n=1}^{\infty} P(\underline{X}^1 \geqslant p^{\underline{\alpha}^{-1}} \underline{x})^n p(1-p)^{n-1} = \frac{p\overline{F}_{\underline{X}}(p^{\underline{\alpha}^{-1}} \underline{x})}{1 - (1-p)\overline{F}_{\underline{X}}(p^{\underline{\alpha}^{-1}} \underline{x})}, \tag{4.12}$$

where $\overline{F}_{\underline{X}}(p^{\underline{\alpha}^{-1}} \underline{x})$ satisfies

$$\overline{F}_{\underline{X}}(p^{\underline{\alpha}^{-1}} \underline{x}) = \frac{1}{1 + \psi(p^{\underline{\alpha}^{-1}} \underline{x})}. \tag{4.13}$$

Substitute (4.13) in (4.12), then

$$\overline{H}_{\underline{m}_p}(\underline{x}) = \frac{\dfrac{p}{1 + \psi(p^{\underline{\alpha}^{-1}} \underline{x})}}{1 - \dfrac{(1-p)}{1 + \psi(p^{\underline{\alpha}^{-1}} \underline{x})}} = \frac{1}{1 + \frac{1}{p}\psi(p^{\underline{\alpha}^{-1}} \underline{x})}. \tag{4.14}$$

In (4.14) and by the definition of $\mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$, the functional equation $\psi(\cdot)$ satisfies

$$\psi(x_1, x_2, \cdots, x_k) = \frac{1}{p}\psi(p^{1/\alpha_1}x_1, p^{1/\alpha_2}x_2, \cdots, p^{1/\alpha_k}x_k) \stackrel{\Delta}{=} \frac{1}{p}\psi(p^{\underline{\alpha}^{-1}}\underline{x}).$$

Hence, Eq. (4.14) is the same as $\overline{H}_{\underline{m}_p}(\underline{x}) = \frac{1}{1+\psi(x_1,\cdots,x_k)} = \overline{F}_{\underline{X}}(\underline{x})$ for all $\underline{x} > \underline{0}$, thus $p^{-\underline{\alpha}^{-1}}\underline{m}_p \stackrel{d}{=} \underline{X}^1$ is followed. $\square$

Theorem 4.1 can be extended to any finite steps of repeated geometric minimization procedure. It is stated as follows:

Suppose we start with a sequence of i.i.d. k-dimensional random vectors with common joint survival function $\overline{F}_1(\cdot)$, i.e., assuming $\underline{X}_1^{(1)}, \underline{X}_2^{(2)}, \cdots, \underline{X}_n^{(2)}, \cdots, i.\underset{\sim}{i}.d.\ \overline{F}_1(\cdot)$, let $N_1 \sim$ geometric (p$_1$), define $\underline{X}_{N_1}^{(1)} = \min\{\underline{X}_1^{(1)}, \underline{X}_2^{(1)}, \cdots, \underline{X}_{N_1}^{(1)}\}$ as the k-dimensional geometric minima of $\{\underline{X}_i^{(1)}\}$, assuming the joint survival function is $\overline{F}_2(\cdot)$.

Also let $\underline{X}_1^{(1)}, \underline{X}_2^{(2)}, \cdots, \underline{X}_n^{(2)}, \cdots$, i.$\underset{\sim}{\text{i}}$.d. $\overline{F}_2(\cdot)$ and N_2 $\sim$geometric (p_2), define $X_{N_2}^{(2)} = \min\{\underline{X}_1^{(2)}, \underline{X}_2^{(2)}, \cdots, \underline{X}_{N_2}^{(2)}\}$, suppose $\underline{X}_{N_2}^{(2)} \sim \overline{F}_3(\cdot)$. In general, for any fixed $\ell = 2, 3, \cdots$, after $(\ell-1)$ steps of repeated geometric minimization procedures, let $\underline{X}_1^{(\ell-1)}, \underline{X}_2^{(\ell-1)}, \cdots, \underline{X}_n^{(\ell-1)}, \cdots$, i.$\underset{\sim}{\text{i}}$.d. $\overline{F}_{\ell-1}(\cdot)$, let $N_{\ell-1}$ $\sim$geometric $(p_{\ell-1})$, define $\underline{X}_{N_{\ell-1}}^{(\ell-1)} = \min\{\underline{X}_1^{(\ell-1)}, \underline{X}_2^{(\ell-1)}, \cdots, \underline{X}_{N_{\ell-1}}^{(\ell-1)}\}$, suppose $\underline{X}_{N_{\ell-1}}^{(\ell-1)} \sim \overline{F}_\ell(\cdot)$

The following theorem characterizes the multivariate-semi-Pareto distribution via any finite steps of repeated geometric minimization.

Theorem 4.2. *Let $\{\underline{X}_1^{(1)}, \underline{X}_2^{(1)}, \cdots, \underline{X}_n^{(1)}, \cdots\}$ be a sequence of i.i.d. non-negative k-variate random vectors with common joint survival function $\overline{F}_1(\cdot)$. For each $\ell = 2, 3, \cdots$, define $\overline{F}_\ell(\cdot)$ sequentially in such manner that $\overline{F}_\ell(\cdot)$ is the joint survival function of a geometric $(p_{\ell-1})$ minima $(0 < p_{\ell-1} < 1)$ of a random sample of $\{\underline{X}_i^{(\ell-1)}\}$ i.$\underset{\sim}{\text{i}}$.d. $\overline{F}_{\ell-1}(\cdot)$, if there exist two parameter vectors $\underline{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_k) > \underline{0}$ as the scale parameter of $\overline{F}_l(\cdot)$ and $\underline{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_k) > \underline{0}$ as the inequality parameter of $\overline{F}_l(\cdot)$, then the following two statements are equivalent:*

(1) *For each finite $\ell = 2, 3, \cdots$,*

$$\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \overline{F}_1(\underline{x}) \tag{4.15}$$

for any $\underline{x} = (x_1, \cdots, x_k) > \underline{0}$, where $(\prod_{j=1}^{\ell-1} p_j)^{\underline{\gamma}}\underline{x} \stackrel{\Delta}{=} \left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma_1} x_1, \cdots, \left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma_k} x_k\right)$, or equivalently,

$$\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{N_{\ell-1}}^{(\ell-1)} \stackrel{d}{=} \underline{X}^{(1)} \sim \overline{F}_1(\cdot). \tag{4.16}$$

(2) *The common joint survival function of* $\{\underline{X}_i^{(1)}, i \geqslant 1\}$, $\overline{F}_l(\cdot)$ *is a* $\mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\gamma}, p)$ *distribution with* $p = \prod_{j=1}^{\ell-1} p_j$.

Proof. (1) $\Rightarrow$ (2). If Eq. (4.15) holds, i.e., $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{N_{\ell-1}}^{(\ell-1)} \stackrel{d}{=} \underline{X}^{(1)}$, for any finite $\ell = 2, 3, \cdots$, the recursive relation among the joint survival functions $\{\overline{F}_\ell(\cdot)\}_{\ell=1}^{\infty}$ is derived by conditioning on the geometric $N_{\ell-1}$ random variable, the joint survival function of the geometric $(p_{\ell-1})$ minima $\underline{X}_{N_{\ell-1}}^{(\ell-1)}$, $\overline{F}_\ell(\cdot)$ is derived as

$$\begin{aligned}\overline{F}_\ell(\underline{x}) &= P(\underline{X}_{N_{\ell-1}}^{(\ell-1)} \geqslant \underline{x}) = \sum_{n=1}^{\infty} P(\min_{1\leqslant i\leqslant n} \underline{X}_i^{(\ell-1)} \geqslant \underline{x}) P(N_{\ell-1} = n) \\ &= \sum_{n=1}^{\infty} (\overline{F}_{\ell-1}(\underline{x}))^n p_{\ell-1}(1-p_{\ell-1})^{n-1} = \frac{p_{\ell-1}\overline{F}_{\ell-1}(\underline{x})}{1-(1-p_{\ell-1})\overline{F}_{\ell-1}(\underline{x})} \end{aligned} \tag{4.17}$$

from Eq. (4.16), it is discerned that for any $\underline{x} > \underline{0}$,

$$P\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{N_{\ell-1}}^{(\ell-1)} \geqslant \underline{x}\right) = P(\underline{X}^{(1)} \geqslant \underline{x}), \quad \text{then } P\left(\underline{X}_{N_{\ell-1}}^{(\ell-1)} \geqslant \left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right), \quad \text{i.e.,}$$

$$\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \overline{F}_1(\underline{x}), \quad \text{and equivalently,}$$

$$\overline{F}_\ell(\underline{x}) = \overline{F}_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{x}\right). \tag{4.18}$$

In Eq. (4.18), express $\varphi_\ell(\underline{x}) \stackrel{\Delta}{=} \frac{1-\overline{F}_\ell(\underline{x})}{\overline{F}_\ell(\underline{x})}$, then $\overline{F}_\ell(\underline{x}) = \frac{1}{1+\varphi_\ell(\underline{x})}$ for each $\ell \geqslant 1$, substitute back to Eq. (4.17), we get

$$\frac{1}{1+\varphi_\ell(\underline{x})} = \frac{p_{\ell-1}\cdot \dfrac{1}{1+\varphi_{\ell-1}(\underline{x})}}{1-(1-p_{\ell-1})\cdot\left(\dfrac{1}{1+\varphi_{\ell-1}(\underline{x})}\right)}, \tag{4.19}$$

after algebraic simplification, we conclude that for all $\underline{x} > \underline{0}$, $\varphi_\ell(\underline{x}) = (p_{\ell-1})^{-1}\varphi_{\ell-1}(\underline{x})$. It follows by iteration that

$$\varphi_\ell(\underline{x}) = \left(\prod_{j=1}^{\ell-1} p_j\right)^{-1} \varphi_1(\underline{x}) \tag{4.20}$$

for all $\ell = 2, 3, \cdots$, from Eq. (4.20), and the expression of $\varphi_\ell(\cdot)$, we have known that $\frac{1}{1+\varphi_\ell(\underline{x})} = \frac{1}{1+\varphi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{x}\right)}$, hence $\varphi_\ell(\underline{x}) = \varphi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{x}\right)$, on the other hand by Eq. (4.20), it follows that the functional equation $\varphi_l(\cdot)$ satisfies $\varphi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{x}\right) = \left(\prod_{j=1}^{\ell-1} p_j\right)^{-1} \varphi_1(\underline{x})$, or equivalently,

$$\varphi_1(\underline{x}) = \left(\prod_{j=1}^{\ell-1} p_j\right) \varphi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{x}\right) \tag{4.21}$$

i.e., for any $\underline{x} = (x_1, x_2, \cdots, x_k) > \underline{0}$, $\varphi_1(\cdot)$ satisfies

$$\begin{aligned}\varphi_1(x_1, x_2, \cdots, x_k) = \left(\prod_{j=1}^{\ell-1} p_j\right) \cdot \varphi_1\Bigg(&\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\gamma_1} x_1,\\ &\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\gamma_2} x_2, \cdots, \left(\prod_{j=1}^{\ell-1} p_j\right)^{-\gamma_k} x_k\Bigg).\end{aligned}$$

Hence for $\varphi_1(\cdot)$, there exists a positive real number $p \underline{\underline{\Delta}} \prod_{j=1}^{\ell-1} p_j$, $(0 < p < 1)$, such that

$$\varphi_1(\underline{x}) = p \cdot \varphi_1(p^{-\underline{\gamma}} \underline{x}). \tag{4.22}$$

Now, referring back to Definition 4.1 and Eq. (4.1), it is observed that the joint survival function of $\overline{F}_1(\cdot)$ can be indeed written as $\overline{F}_1(\underline{x}) = \frac{1}{1+\varphi_1(\underline{x})}$ with $\varphi_1(\underline{x}) = p \cdot \varphi_1(p^{-\underline{\gamma}} \underline{x})$. Thus for all $i \geqslant 1$, $\underline{X}_i^{(1)} \sim \mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\gamma}, p)$ follows. (Note that the $\underline{\gamma}$ and $\underline{\alpha}$ in the definition are reciprocal to each other).

(2) $\Rightarrow$ (1). On the other hand, if $\underline{X}_i^{(1)} \sim \mathrm{MSP}^{(k)}(\underline{\sigma}, \underline{\gamma}, p)$ with $p = \prod_{j=1}^{\ell-1} p_j$, then according to Definition 4.1, $\overline{F}_1(\cdot)$ can be written as

$$\overline{F}_1(\underline{x}) = \frac{1}{1+\psi_1(\underline{x})} \quad \text{with} \quad \psi_1(\underline{x}) = p \cdot \psi_1(p^{-\underline{\gamma}} \underline{x}), \tag{4.23}$$

also, the joint survival function of the scaled geometric minima $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{N_{\ell-1}}^{(\ell-1)}$ is for any $\underline{x} \geqslant \underline{0}$,

$$P\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{N_{\ell-1}}^{(\ell-1)} \geqslant \underline{x}\right) = P\left(\underline{X}_{N_{\ell-1}}^{(\ell-1)} \geqslant \left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)$$

refer back to Eq. (4.17), then

$$\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \frac{p_{\ell-1}\,\overline{F}_{\ell-1}\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)}{1-(1-p_{\ell-1})\overline{F}_{\ell-1}\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)} \tag{4.24}$$

use the same expression for any $\ell \geqslant 1$, $\varphi_\ell(\underline{x}) \stackrel{\Delta}{=} \frac{1-\overline{F}_\ell(\underline{x})}{\overline{F}_\ell(\underline{x})}$, and thus $\overline{F}_\ell(\underline{x}) = \frac{1}{1+\varphi_\ell(\underline{x})}$, substituting this in Eq. (4.24), we have $\varphi_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \frac{1}{p_{\ell-1}}\varphi_{\ell-1}\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)$. It follows by iteration that $\varphi_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \frac{1}{\left(\prod_{j=1}^{\ell-1} p_j\right)}\varphi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)$, note $\varphi_l(\cdot)$ is the expression $\varphi_l(\underline{x}) \stackrel{\Delta}{=} \frac{1-\overline{F}_l(\underline{x})}{\overline{F}_l(\underline{x})}$, and then $\overline{F}_l(\underline{x}) = \frac{1}{1+\varphi_l(\underline{x})}$, comparing this with Eq. (4.23), so $\varphi_l(\underline{x}) \equiv \psi_l(\underline{x})$, so the functional equation $\varphi_1(\cdot)$ satisfies $\varphi_1(\underline{x}) = p\cdot\varphi_1(p^{-\underline{\gamma}}\underline{x})$, or equivalently, $\varphi_1(\underline{x}) = \frac{1}{p}\varphi_1(p^{\underline{\gamma}}\underline{x})$ with $p = \prod_{j=1}^{\ell-1} p_j$, then Eq. (4.21) becomes

$$\frac{1}{1+\varphi_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)} = \frac{1}{1+\frac{1}{\left(\prod_{j=1}^{\ell-1} p_j\right)}\varphi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right)} = \frac{1}{1+\varphi_1(\underline{x})} = \overline{F}_1(\underline{x}).$$

Therefore, $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{N_{\ell-1}}^{(\ell-1)} \stackrel{d}{=} \underline{X}^{(1)}$, hence the proof is complete. □

Remark. If the common distribution $\overline{F}_l(\cdot)$ of $\underline{X}^{(1)}$ is in the family of $F_{\underline{\sigma},\underline{\gamma}}$, where $\underline{\sigma} = (\sigma_1, \cdots, \sigma_k) > \underline{0}$, $\underline{\gamma} = (\gamma_1, \cdots, \gamma_k) > \underline{0}$ and $F_{\underline{\sigma},\underline{\gamma}}$ denotes the family of all distribution function $F(\cdot)$ with the property that the local behavior of $\overline{F}(\cdot)$ nears its lower bound $\underline{x} \to \underline{0}$ is approximated by the asymptotic form $\overline{F}(\underline{x}) \sim 1-\sum_{i=1}^{k}(x_i/\sigma_i)^{1/\gamma_i}$, then there are two limit theorems for characterizing Arnold's [2] MP$^{(k)}$(III) ($\underline{0}$, $\underline{\sigma}$, $\underline{\gamma}$) distribution which had been proved in Yeh's [11] Theorems 4.1 and 4.2.

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